

## Vertex operator for the ultradiscrete KdV equation

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## FAST TRACK COMMUNICATION

**Vertex operator for the ultradiscrete KdV equation****Yoichi Nakata**

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Online at [stacks.iop.org/JPhysA/42/412001](http://stacks.iop.org/JPhysA/42/412001)**Abstract**

We propose an ultradiscrete analogue of the vertex operator in the case of the ultradiscrete KdV equation, which maps  $N$ -soliton solutions to  $N + 1$ -soliton ones.

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The ultradiscrete KdV equation (udKdVeq)

$$F_j^t + F_{j+1}^{t+2} = \max(F_j^{t+2} + F_{j+1}^t - 2R, F_j^{t+1} + F_{j+1}^{t+1}) \quad (R \geq 0) \quad (1)$$

was first proposed in [6] to describe the dynamics of the box and ball system (BBS) [5], which is a cellular automaton with soliton-like behaviour. It has a rich structure, including the existence of  $N$ -soliton solutions and an infinite amount of conserved quantities [3] like most ordinary soliton equations.

A procedure to obtain a new solution of a soliton equation from a given one is known under the name of Bäcklund transformation (BT) and is generally expressed in the form of differential equations. The ultradiscrete analogue of the BT for the KdV equation in the case of the udKdVeq is presented in [2].

The vertex operator is an operator representation of a BT and was first presented in [1] to describe the structure of the solutions and the symmetries of soliton systems. It maps  $N$ -soliton solutions to  $N + 1$ -soliton ones and all soliton solutions can in fact be generated by repeated application for the vertex operator.

In this paper, we propose an ultradiscrete analogue of the vertex operator in the case of the ultradiscrete KdV equation and prove that the functions generated by this operator indeed solve the udKdVeq.

First, we define the vertex operator and consider the functions generated by this operator.

**Definition 1.** *The multi-variable function  $F$  and the vertex operator  $X$  are defined as follows.*

(i) *The null variable function  $F(;)$  is defined as*

$$F(; ) = 0. \quad (2)$$

(ii) The vertex operator  $X$  depends on one real parameter  $\Omega_1 \geq 0$  and a function  $\eta_1(\Omega_1)$  of  $\Omega_1$ , and maps the null variable function  $F(; )$  to

$$X(\Omega_1, \eta_1)F(; ) := \max(-\eta_1, \eta_1). \quad (3)$$

Accordingly, the one variable function  $F(\Omega_1; \eta_1)$  is defined as

$$F(\Omega_1; \eta_1) := X(\Omega_1, \eta_1)F(; ). \quad (4)$$

(iii) For general  $N \geq 1$ , the vertex operator  $X$  maps an  $N$  variable function  $F(\Omega_1, \dots, \Omega_N; \eta_1, \dots, \eta_N)$  (written as  $F(\Omega; \eta)$  for brevity) to the  $N+1$  variable function as

$$\begin{aligned} X(\Omega_{N+1}, \eta_{N+1})F(\Omega; \eta) &:= \max(-\eta_{N+1} + F(\Omega; \eta + \min_{\Omega_{N+1}}(\Omega)), \\ &\quad \eta_{N+1} + F(\Omega; \eta - \min_{\Omega_{N+1}}(\Omega))) \\ &=: F(\Omega_1, \dots, \Omega_N, \Omega_{N+1}; \eta_1, \dots, \eta_N, \eta_{N+1}), \end{aligned} \quad (5)$$

where

$$\min_{\Omega_{N+1}}(\Omega) = {}^t(\min(\Omega_1, \Omega_{N+1}), \dots, \min(\Omega_N, \Omega_{N+1})). \quad (6)$$

The parameters  $\Omega$  and the functions  $\eta$  in the vertex operator  $X$  are in fact the amplitude and phase parameter of the new soliton, inserted by the operator. Definition (5) indicates that all pre-existing solitons described by  $F$  have their phases shifted by inserting a new soliton.

Here, we present the basic properties of the operator  $X$  and the corresponding function  $F$ .

**Proposition 1.** *The action of the vertex operators is commutative.*

**Proof.** We calculate  $X(\Omega_b, \eta_b)X(\Omega_a, \eta_a)F(\Omega; \eta)$  directly following definition (5):

$$\begin{aligned} X(\Omega_b, \eta_b)X(\Omega_a, \eta_a)F(\Omega; \eta) &= \max(-\eta_a - \eta_b - \min(\Omega_a, \Omega_b) + F(\Omega; \eta + \min_{\Omega_a}(\Omega) + \min_{\Omega_b}(\Omega)), \\ &\quad \eta_a - \eta_b + \min(\Omega_a, \Omega_b) + F(\Omega; \eta - \min_{\Omega_a}(\Omega) + \min_{\Omega_b}(\Omega)), \\ &\quad -\eta_a + \eta_b + \min(\Omega_a, \Omega_b) + F(\Omega; \eta + \min_{\Omega_a}(\Omega) - \min_{\Omega_b}(\Omega)), \\ &\quad \eta_a + \eta_b - \min(\Omega_a, \Omega_b) + F(\Omega; \eta - \min_{\Omega_a}(\Omega) - \min_{\Omega_b}(\Omega))). \end{aligned} \quad (7)$$

From this relation it is clear that interchanging the subscripts  $a$  and  $b$  does not change its overall value.  $\square$

Rewriting this proposition in the language of the function  $F$  yields the following corollary.

**Corollary 1.** *The  $N$  variable functions  $F(\Omega; \eta)$  are invariant under the permutation of their parameters, i.e.*

$$F(\Omega_1, \dots, \Omega_N; \eta_1, \dots, \eta_N) = F(\Omega_{\sigma(1)}, \dots, \Omega_{\sigma(N)}; \eta_{\sigma(1)}, \dots, \eta_{\sigma(N)}) \quad (\sigma \in \mathfrak{S}_N). \quad (8)$$

Next, let us prove that these functions are indeed solutions of the udKdVeq by means of the recursive form (5). This property indicates that the vertex operator defined in (5) is nothing but the operator, well known from soliton theory, which maps an  $N$ -soliton solution to an  $N + 1$ -soliton solution.

**Theorem 1.** *The  $N$  variable function  $F(\Omega; \eta)$  solves the udKdVeq (1) for*

$$\eta = \mathbf{C} - j\mathbf{K} + t\Omega, \quad (9)$$

where

$$\mathbf{K} = \min_R(\Omega) \tag{10}$$

and  $C$  is a constant.

**Proof.** By virtue of corollary 1, we can fix the labels of the parameters

$$\Omega_N \geq \Omega_{N-1} \geq \dots \geq \Omega_1 \geq 0 \tag{11}$$

without loss of generality. By virtue of this ordering, the phase shifts in definition (5) simplify to

$$\min(\Omega_i, \Omega_N) = \Omega_i \quad (i = 1, \dots, N - 1). \tag{12}$$

It should be noted that the phase shifts  $\eta \rightarrow \eta \pm \Omega$  are equivalent to the time shifts  $t \rightarrow t \pm 1$ . The  $N$  variable function  $F(\Omega; \eta)$  is then reduced to  $F_j^{(N),t}$ , which is written as

$$F_j^{(N),t} = \max \left( -C_N + jK_N - t\Omega_N + F_j^{(N-1),t+1}, \right. \\ \left. C_N - jK_N + t\Omega_N + F_j^{(N-1),t-1} \right) \quad (N \geq 1) \tag{13}$$

$$F_j^{(0),t} = 0 \quad (N = 0). \tag{14}$$

We shall prove the theorem inductively. It is clear that  $F_j^{(0),t}$  solves equation (1) because of the positivity of  $R$ . Now, let us assume that the theorem holds for  $N - 1$ . By substituting (13) in the udKdVeq (1), the left-hand side can be rewritten as

$$F_j^{(N),t} + F_{j+1}^{(N),t+2} = \max \left( -2C_N + (2j + 1)K_N - (2t + 2)\Omega_N + F_j^{(N-1),t+1} + F_{j+1}^{(N-1),t+3}, \right. \\ \left. 2C_N - (2j + 1)K_N + (2t + 2)\Omega_N + F_j^{(N-1),t-1} + F_{j+1}^{(N-1),t+1}, \right. \\ \left. K_N - 2\Omega_N + F_j^{(N-1),t-1} + F_{j+1}^{(N-1),t+3}, \right. \\ \left. -K_N + 2\Omega_N + F_j^{(N-1),t+1} + F_{j+1}^{(N-1),t+1} \right). \tag{15}$$

In this expression it looks as if the maximum in (15) has four arguments. However, the third argument cannot be the maximum because it is always less than the fourth one by virtue of the following lemma, in case  $l = -1, m = 2$ .  $\square$

**Lemma 1.** *Let*

$$H_j^{(N),t} = F_j^{(N),t+m+2} - F_{j+l}^{(N),t+2} - F_j^{(N),t+m} + F_{j+l}^{(N),t}, \tag{16}$$

where the parameters  $l$  and  $m$  satisfy  $lK_N + m\Omega_N \geq 0$ . Then the relation

$$H_j^{(N),t} \leq 2(lK_N + m\Omega_N) \tag{17}$$

holds.

**Proof.** To prove this lemma, let us introduce some fundamental properties of the max operator.  $\square$

**Proposition 2.** *The inequalities*

$$\max(a + c, b + d) \leq \max(a, b) + \max(c, d) \tag{18}$$

$$\max(x, y) - \max(z, w) \leq \max(x - z, y - w) \tag{19}$$

hold for arbitrary  $a, b, c, d, x, y, z, w \in \mathbb{R}$

**Proof.** Proof of (18): the right-hand side of equation (18) can be expanded to yield  $\max(a + c, a + d, b + c, b + d)$ , which includes all candidates of the left-hand side. Proof of (19): put  $a = z, b = w, c = x - z, d = y - w$  in (18) respectively.  $\square$

If we now apply

$$\min(a, b) = -\max(-a, -b), \tag{20}$$

we obtain the following corollary.

**Corollary 2.** *The above inequalities (18), (19) hold even when  $\max$  and  $\leq$  are replaced by  $\min$  and  $\geq$ .*

By employing inequality (19), we obtain

$$F_j^{(N),t+m+2} - F_{j+l}^{(N),t+2} \leq \max(-lK_N - m\Omega_N + F_j^{(N-1),t+m+3} - F_{j+l}^{(N-1),t+3}, \\ lK_N + m\Omega_N + F_j^{(N-1),t+m+1} - F_{j+l}^{(N-1),t+1}) \tag{21}$$

$$F_{j+l}^{(N),t} - F_j^{(N),t+m} \leq \max(lK_N + m\Omega_N + F_{j+l}^{(N-1),t+1} - F_j^{(N-1),t+m+1}, \\ -lK_N - m\Omega_N + F_{j+l}^{(N-1),t-1} - F_j^{(N-1),t+m-1}). \tag{22}$$

Adding the inequalities yields

$$H_j^{(N),t} \leq \max(H_j^{(N-1),t+1}, 2(lK_N + m\Omega_N), -2(lK_N + m\Omega_N) + H_j^{(N-1),t+1} + H_j^{(N-1),t-1}, \\ H_j^{(N-1),t-1}). \tag{23}$$

By taking into account the relation  $\Omega_N \geq \Omega_{N-1}$ , it can be shown inductively that the four arguments in this maximum are all less than  $2(lK_N + m\Omega_N)$ .

By means of the same procedure, the right-hand side of equation (1) is rewritten as

$$\max(F_j^{t+2} + F_{j+1}^t - 2R, F_j^{t+1} + F_{j+1}^{t+1}) = \max(-2C_N - (2j + 1)K_N - (2t + 2)\Omega_N \\ + \max(F_j^{(N-1),t+3} + F_{j+1}^{(N-1),t+1} - 2R, F_j^{(N-1),t+2} + F_{j+1}^{(N-1),t+2}), \\ 2C_N + (2j + 1)K_N + (2t + 2)\Omega_N \\ + \max(F_j^{(N-1),t+1} + F_{j+1}^{(N-1),t-1} - 2R, F_j^{(N-1),t} + F_{j+1}^{(N-1),t+1}), \\ \max(K_N + 2\Omega_N + F_j^{(N-1),t+1} + F_{j+1}^{(N-1),t+1} - 2R, \\ K_N + F_j^{(N-1),t} + F_{j+1}^{(N-1),t+2})). \tag{24}$$

There are three arguments in the principal maximum. The first and second arguments are the same as those on the left-hand side because  $F_j^{(N-1),t}$  solves equation (1) by assumption. It can also be shown that the last argument is also the same by employing the method which was used to prove lemma 1.

The condition which expresses the equality of the last argument to that of the left-hand side then reduces to:

$$0 = \max(2(K_N - R), 2(K_N - \Omega_N) + F_j^{(N-1),t} + F_{j+1}^{(N-1),t+2} - F_j^{(N-1),t+1} - F_{j+1}^{(N-1),t+1}) \tag{25}$$

and holds by virtue of the following lemma.

**Lemma 2.** *Let*

$$\tilde{H}_j^{(N),t} = F_j^{(N),t} + F_{j+1}^{(N),t+2} - F_j^{(N),t+1} - F_{j+1}^{(N),t+1}. \tag{26}$$

One then has

$$0 \leq \tilde{H}_j^{(N),t} \leq 2(\Omega_N - K_N). \tag{27}$$

**Proof.** By employing the inequality (19), we obtain

$$F_j^{(N),t} - F_{j+1}^{(N),t+1} \leq \max(-K_N + \Omega_N + F_j^{(N-1),t+1} - F_{j+1}^{(N-1),t+2}, K_N - \Omega_N + F_j^{(N-1),t-1} - F_{j+1}^{(N-1),t}) \tag{28}$$

$$F_{j+1}^{(N),t+2} - F_j^{(N),t+1} \leq \max(K_N - \Omega_N + F_{j+1}^{(N-1),t+3} - F_j^{(N-1),t+2}, -K_N + \Omega_N + F_{j+1}^{(N-1),t+1} - F_j^{(N-1),t}). \tag{29}$$

Adding these inequalities, we obtain

$$\begin{aligned} \tilde{H}_j^{(N),t} \leq \max & \left( \tilde{H}_j^{(N-1),t+1}, 2(\Omega_N - K_N) - \tilde{H}_j^{(N-1),t}, \right. \\ & \left. - 2(\Omega_N - K_N) + \tilde{H}_j^{(N-1),t-1} + H_{j+1}^{(N-1),t+1} \Big|_{l=-1, m=1}, \right. \\ & \left. \tilde{H}_j^{(N-1),t-1} \right), \end{aligned} \tag{30}$$

where  $H_j^{(N),t}$  is given by lemma 1. It can be shown inductively that all arguments are less than  $2(\Omega_N - K_N)$ .

The positivity of  $\tilde{H}_j^{(N),t}$  is proven by  $F_j^{(N-1),t}$  satisfying udKdVeq (1). □

Finally, we give an explicit example of the action of the vertex operator  $X$ , in the case of the BBS. The BBS corresponds to the udKdVeq (1) by means of the dependent variable transformation

$$B_j^t = \frac{1}{2} (F_j^{t+1} + F_{j+1}^t - F_{j+1}^{t+1} - F_j^t) \tag{31}$$

and is expressed as

$$B_n^{t+1} = \min \left( R - B_n^t, \sum_{n=-\infty}^{j-1} (B_n^{t+1} - B_n^t) \right). \tag{32}$$

The BBS is required to satisfy the following boundary conditions:  $B_j^t = 0$  for  $j \ll 0$ . Under these conditions, it can be shown that the possible values of the variable  $B_j^t$  that correspond to the  $N$ -soliton solutions of theorem 1 (i.e. constructed by means of the vertex operator  $X$ ) are restricted to the set  $\{0, 1\}$ .

In figure 1 we depict the  $j$ -lattice by a row of boxes, containing a ball when  $B_j^t$  is equal to 1, and empty whenever  $B_j^t = 0$ . The lattices depicted in the figure are the vacuum lattice, and the lattices obtained from vacuum by successive application of the vertex operator  $X$  with parameter values  $(\Omega_1, C_1) = (1, 0)$ ,  $(\Omega_2, C_2) = (3, -2)$  and  $(\Omega_3, C_3) = (2, 0)$ .

In this paper, we proposed an ultradiscrete analogue of the vertex operator for the ultradiscrete KdV equation and discussed its properties. We presented a recursive representation of the  $N$ -soliton solution generated by this operator and proved that it indeed solves the udKdVeq.

In [4], a different technique for describing the udKdV  $N$ -soliton solutions was introduced, relying on signature-free Casorati-type determinants or permanents. Although these permanents describe the same solutions (already proposed in [6]) as those constructed by recursive application of the ultradiscrete vertex operator, we would like to stress that the two approaches are quite different. In fact, the vertex operator approach is closely related to the existence of certain symmetry algebras for integrable systems and the exact relation of our

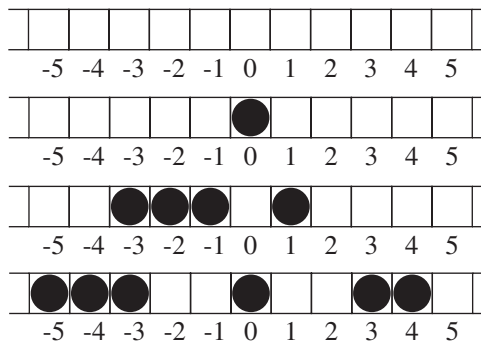


Figure 1. BBS obtained by successive application of the vertex operator.

ultradiscrete operator to the symmetries of ultradiscrete systems is an especially interesting problem we want to address in the future. It would also be interesting to find vertex operators for other ultradiscrete soliton equations, such as for example the ultradiscrete analogue of the KP equation.

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